Diffusion in stochastically perturbed Hamiltonian systems with applications to the recent LHC dynamic aperture experiments

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In this paper, we review diffusion phenomena in stochastically perturbed Hamiltonian systems, with the aim of defining the framework to use Nekhoroshev-like estimates as prototypes for the form of the diffusion coefficient. A discussion of the features of this framework is carried out. More importantly, the results of numerical simulations based on the proposed models are compared against the experimental data from recent measurements performed at the CERN Large Hadron Collider (LHC) of the extent of phase space where bounded motions occur. The main conclusions are presented and discussed in detail together with future steps.

Keywords: Non-linear beam dynamics, diffusion process, Nekhoroshev estimates, dynamic aperture

1. Introduction

The study of diffusion in the phase space of non-integrable Hamiltonian systems is a very difficult problem due to the extreme sensitive dependence of the orbit evolution on the initial conditions. The phenomenon of Arnold diffusion [1] is generic in Hamiltonian systems with two or more degrees of freedom, but its relevance in the applications is still debated. Indeed, any real macroscopic physical system cannot realise the symplectic deterministic character of the dynamics at arbitrary small scales due to the unavoidable presence of external random perturbations that destroy the time-coherence in the orbits' evolution. Nevertheless some results of perturbation theory turn out to be robust with respect to perturbations of the system under consideration and they can provide effective laws in the study of the stability problem. Nekhoroshev's theorem [2] is among these results and the corresponding estimate for the orbit stability time has been applied in many fields, from celestial mechanics to accelerator physics. In this paper, we focus on the problem of dynamic aperture estimate in beam dynamics. The dynamic aperture (DA) is the amplitude of the phase space region where stable motion occurs. It is one of the key quantities for the design of modern colliders based on superconducting magnets, such as Tevatron [3–5], HERA [6–9], RHIC [10], the Superconducting Super Collider

(SSC)[11,12], and LHC (see e.g., Ref.[13] for a detailed overview).

The concept of stable motion needs a proper definition of the time frame. In a mathematical sense, stable motion implies bounded motion for arbitrary time. In a physical context, particle stability can be linked to a maximum number of turns $N_{\max}$ for which bounded motion occurs, where $N_{\max}$ is set on the basis of the specific application under consideration. If an ensemble of initial conditions defined on a polar grid ($x = r\,\cos\theta\,,\ y = r\,\sin\theta\ \ 0 \leq \theta \leq \pi/2$, where x, y are expressed in units σ_x, σ_y of the beam dimension) is tracked for up to $N_{\max}$ turns to assess their stability, then the DA can be defined as[14]:

$$D(N) = \frac{2}{\pi}\int_0^{\pi/2} r(\theta; N)\, d\,\theta \equiv \langle r(\theta; N)\rangle\,. \tag{1}$$

where $r(\theta; N)$ stands for the last stable amplitude (disregarding any stable domain disconnected from the origin) for up to N turns in the direction θ. In this way, dynamic aperture can be considered a function of time, with an asymptotic value representing the region of stability for arbitrary time.

An accurate numerical computation of DA, as well as a good estimate of the error associated with the protocol used in the numerical simulations, is of paramount importance to ensure the reliability of DA as a figure-of-merit for assessing synchrotron performance. A general discussion of the DA definition, its computation, and accuracy can be found, e.g., in Ref.[14]. Computation consists of simulating the evolution of a large number of initial conditions, distributed to provide good coverage of the phase space under study, probing whether motion remains bounded over the time interval selected for the simulations.

Given the CPU-intense character of these simulations, studies have explored models to fit, and eventually extrapolate, the dependence of the DA on the number of turns[15,16] have been looked for. The rationale is that long-term behaviour of the DA, a computationally heavy task, can be extrapolated using knowledge from numerical simulations performed over a smaller number of turns. Potentially, a large number of initial conditions can be used to improve the accuracy of numerical simulations. Increasing the number of initial conditions has no drawbacks in terms of CPU-time needed, as parallelisation over the initial conditions can be easily performed[17]. Additionally, a more efficient estimate of the long-term behaviour of the DA would expedite analysis of several configurations of the circular accelerator, which is sometimes obligatory to gain insight in the deeper nature of the beam dynamics.

The answer to the quest for a model for the time-evolution of DA was provided by two fundamental results of the theory of dynamical systems, namely the Kolmogorov-Arnold-Moser (KAM)[18] and the Nekhoroshev[2] theorems. According to the results of Refs.[15,16], the following scaling law holds

$$D(N) = D_\infty + \frac{b}{(\log N)^\kappa}\,, \tag{2}$$

where D_∞ represents the asymptotic value of the amplitude of the stability domain. b and κ are additional parameters.

The model (2) is compatible with the hypothesis that the phase space is partitioned into two regions: a central core, with $r < D_\infty$, where KAM[18] surfaces bound the motion, thus producing a stable behaviour apart from a set of small measure where Arnold diffusion can take place; and an outer part, with $r > D_\infty$, where the escape rate to infinity is given by a Nekhoroshev-like estimate[2,19,20]

$$N(r) \, = \, N_0 \exp\left[\left(\frac{r_*}{r}\right)^{1/\kappa}\right] \tag{3}$$

where $N(r)$ is the number of turns that are estimated to be stable for particles with initial amplitude smaller than r. Experience with the analysis of data from numerical simulations of various configurations of the LHC[16] and from experimental data from the Tevatron[21] showed that the fit parameters b, κ, D_∞ can assume signs that go beyond that predicted by strictly applying the model based on Nekhoroshev theorem. At this point, the scaling law for DA has been used to propose a model for the evolution of beam intensity in a hadron synchrotron[21], which is the basis of the novel experimental method proposed in this article to measure DA. If the beam distribution is Gaussian in x and y

$$\rho_G(x,y) = \frac{1}{2\,\pi\,\sigma_x\,\sigma_y}\, e^{-\left(\frac{x^2}{2\,\sigma_x^2}+\frac{y^2}{2\,\sigma_y^2}\right)} \tag{4}$$

then after transforming to polar co-ordinates and applying (1), i.e., assuming that particles with amplitude beyond $D(N)$ at turn N are lost, then the evolution of beam intensity N_b can be found as

$$\frac{N_\mathrm{b}(t)}{N_\mathrm{b}(1)} = 1 - \int_{D(t)}^{+\infty} e^{-\frac{r^2}{2}}\, r\, dr = 1 - e^{-\frac{D^2(N)}{2}}, \tag{5}$$

where $D(N) \xrightarrow[N\to 1]{} +\infty$, and $D(N)$ is assumed to be expressed in units of sigma. Equation (5) is the basis for establishing a direct link between DA and beam lifetime in a synchrotron.

In this paper an alternative approach is explored, based on the possibility to describe the beam dynamics by means of a Hamiltonian system under the influence of stochastic perturbations. The goal is to build a diffusion equation whose solution represents the evolution of the beam distribution, possibly including also boundary conditions, such as absorbing boundaries. Such an approach features a number of interesting aspects, in particular that of providing a natural description of the beam dynamics in the presence of a collimation system, which is a typical situation in most of modern colliders based on superconducting magnets. The ultimate goal is to set up a general framework to justify the approach based on a diffusive equation as well as to provide observables the can be compared with what is observed in beam experiments.

In fact, in recent years, with the advent of the LHC and the approval of its high-luminosity upgrade[22] the topic of measuring with beam the DA has regained interest, after a break between the design phase of the LHC (see, e.g., Refs.[23–25] and Ref.[26] for a review of the comparison between measurements and simulations), and its commissioning and following operation periods. Indeed, some experimental sessions have been performed in the LHC and two methods have been used to probe the extent of the DA (see Refs.[27–30]) and in this paper the main results from Refs.[28,29] are reviewed in light of the new proposed approach.

The plan of the paper is the following: in section 2 the theory of diffusion processes in stochastically perturbed Hamiltonian system is reviewed, while in section 3 the main observable is derived for later use. In section 4 the experimental technique is described and in section 5 the comparison between the approach based on the diffusion equation and the experimental results is carried out. Finally, some conclusions are drawn in section 6.

2. Diffusion in Hamiltonian Systems

Quasi-integrable Hamiltonian systems are characterized by the existence of a set of invariant tori (the so-called KAM tori[18]) of large measure. Moreover, for multi-dimensional systems, transport in phase space occurs only over a set of initial conditions of extremely small measure, through the topological mechanism of Arnold's diffusion[1]. In any event, the timescale of finite transport phenomena can be extremely long.

In the case of a system described by a symplectic polynomial map with an elliptic fixed point at the origin, the KAM invariant tori exist in a neighbourhood of the fixed point, but their measure decreases as the distance from the fixed point increases, and a weakly-chaotic region takes the place of the broken tori. Indeed, for a given time T, it is possible to identify a neighbourhood of the origin $U(T)$ such that all orbits whose initial conditions belong to $U(T)$ are bounded for all times $t \leq T$.

The concept of DA defines the region where KAM theory applies, and the measure of the set of chaotic orbits is negligible so that Arnold's diffusion is the only effective transport mechanism. Beyond the DA one expects the appearance of a large, weakly-chaotic region, likely generated by the overlap of several non-linear resonances, where the orbits can be trapped for a long, but not extremely long, time before a fast escape to infinity occurs.

The amplitude of the chaotic region depends on the non-linear terms in the map and, according to perturbation theory, the action variables are almost conserved in this region, whereas the angle variables follow a dynamics governed by stochastic fluctuations over a characteristic timescale. In the chaotic region, the diffusive behaviour can be extremely complex, due to the underlying geometrical structures associated with the Hamiltonian character of phase space[31].

Nevertheless, in real situations, like applications to beam dynamics, the presence of external random perturbations cannot be avoided[32], thus creating a homogeneous region in phase space. Under these conditions, one has the possibility of describing the orbit diffusion by means of a stochastically perturbed Hamiltonian, which has been studied in the literature[33–36]. The proposed models have the form

$$H(\theta, I, t) = H_0(I) + \epsilon\, \xi(t)\, H_1(\theta, I) \tag{6}$$

where (θ, I) are action-angle variables and $\xi(t)$ is a regular, stationary stochastic noise with zero mean value and unitary variance that mimics the effect of the chaotic dynamics. Note that in Eq. (6) the parameter ϵ defines the diffusion timescale $\epsilon^2 t$. The stochastic phase flow associated with the canonical equations has the symplectic character due to the regularity of the noise realisations and we assume a fast-decaying correlation function for the noise $\langle \xi \rangle$

$$\langle \xi(t)\xi(t+T) \rangle \simeq \gamma \exp(-\gamma |T|) \tag{7}$$

where γ^{-1} is the correlation timescale. We recall that an exponential decaying of the correlation time is consistent with the existence of positive Ljapounov exponents[31].

Due to the sensitive dependence on the initial conditions, each realisation of the noise simulates the evolution of a different orbit in phase space. In this framework, we look for a statistical description of the evolution of a distribution function $\rho(\theta, I, t)$ for a time of the order of the diffusion timescale, which has to be much longer than the correlation timescale.

The perturbation term $H_1(\theta, I)$ measures the effect linked with the random noise, taking into account phase space inhomogeneities, but the fast escape to infinity of an orbit cannot be described by the stochastic model (6) and one has to introduce an absorbing barrier at a given distance from the fixed point. The evaluation of the long-term dynamic aperture is carried out using the stochastic model (6) with $\epsilon \ll 1$ in terms of the action threshold $I_{\rm da}(t_*)$ for which the probability that a particle with $I_0 \le I_{\rm da}$ is lost at the absorbing boundary for $t \le t_*$ is less than an *a priori* given small, albeit not zero value. In the limit $t_* \to \infty$ all the particles will be lost unless the perturbation vanishes at a finite distance from the origin. This would introduce another boundary condition to the stochastic model.

Perturbation theory suggests a possible estimate for the norm of $H_1(\theta, I)$ based on the asymptotic character of the perturbation series. In fact, for the case of a symplectic polynomial map, provided there are no dominating low-order resonance in phase space, a generic estimate of the reminder of the Birkhoff's Normal Form gives[19]

$$\|R_n(I)\| \propto (n!)^\eta \left(\frac{I}{I_*}\right)^{n/2} . \tag{8}$$

The factorial term takes into account the number of contributions due to the structure of the functional equations defining the perturbation series, the exponent η can be related to the number of degrees of freedom, whereas the parameter I_* is related to the strength of the non-linear terms and can define the amplitude beyond which fast escape to infinity occurs. For each I there exists an optimal order for the Normal Form remainder defined by the relations

$$n^\eta = \left(\frac{I}{I_*}\right)^{1/2} \qquad \Rightarrow \qquad n = \left(\frac{I_*}{I}\right)^{1/2\eta} . \tag{9}$$

By substituting the relation (9) into Eq. (8) one obtains the well-known Nekhoroshev-like estimate

$$\|R_{\rm opt}(I)\| \propto \exp\left[-\eta\left(\frac{I_*}{I}\right)^{1/2\eta}\right] \tag{10}$$

that shows how the optimal estimate scales as a function of the action I. The existence of an optimal remainder for the perturbation series is a fingerprint of the non-integrability of the dynamics and it could be used as a measure of the long-term stability of the orbits.

According to this picture we assume that the Nekhoroshev's estimate gives also a measure of the orbits diffusion in phase space and we study the diffusion in the stochastic models (6) assuming the estimate

$$\|H_1(\theta, I)\| \simeq \exp\left[-\left(\frac{I_*}{I}\right)^{\alpha}\right] . \tag{11}$$

To derive a diffusion equation for the action variables one considers the stochastic Liouville equation[37,38] for the distribution function $\rho(\theta, I, t)$

$$\frac{\partial \rho}{\partial t} + \Omega(I)\frac{\partial \rho}{\partial \theta} + \epsilon\,\xi(t)\left[\frac{\partial H_1}{\partial I}\frac{\partial}{\partial \theta} - \frac{\partial H_1}{\partial \theta}\frac{\partial}{\partial I}\right]\rho = 0 \tag{12}$$

associated to the Hamiltonian system (6) and where $\Omega(I) = dH_0/dI$. Moreover, one needs to perform an averaging procedure over the noise realisations and the angle variables[32,39]. This approach can be justified for $\epsilon \ll 1$ when we are able to distinguish four timescales: the noise decorrelation timescale γ^{-1}; the averaging timescale $T \gg \gamma^{-1}$ at which the noise $\xi(t)$ is well approximated by a white noise; the angle relaxation timescale $\propto \epsilon^{-4/3} \gg T$[40]; the action diffusion timescale $\propto \epsilon^{-2}$[32]. In the limit $\epsilon \to 0$ and $T \to \infty$ with γ finite and $\epsilon^2 T \ll 1$, it is possible to show that the average solution of the stochastic Liouville equation (12) is well approximated by the solution of the Fokker-Planck equation[33,34]

$$\frac{\partial \rho}{\partial \tau} = \frac{1}{2}\frac{\partial}{\partial I}\left(h^2(I)\frac{\partial \rho}{\partial I}\right) \tag{13}$$

where the diffusion coefficient is computed according to

$$h^2(I) = \left\langle \left(\frac{\partial H_1}{\partial \theta}\right)^2 \right\rangle_\theta$$

and τ is a *slow effective time* $\propto \epsilon^2\sqrt{\gamma^{-1}T}\,t$ [a]. The condition $\lim_{I\to 0} h(I) = 0$ produces a natural boundary at $I = 0$ and we introduce an absorbing boundary

[a]As a matter of fact τ has the dimension $[t]^2$

condition at $I = I_{\mathrm{a}}$ representing the position of the fast escape to infinity or that of a collimator.

3. Estimate of losses at the absorbing barrier

According to our assumptions, Eq. (13) describes the orbit diffusion of a weakly-chaotic Hamiltonian system if the perturbation term in (6) is proportional to the optimal remainder of the perturbation series, which measures the non-integrability of the system. We use a Nekhoroshev-like estimate for the stochastic perturbation term in (6) so that the diffusion coefficient has the form

$$h^2(I) = E_*^2 \exp\left[-2\left(\frac{I_*}{I}\right)^\alpha\right] . \tag{14}$$

The parameter I_* is related to the apparent radius of convergence of the asymptotic perturbative series and it could be used to measure the non-linear effects, the exponent α is usually related to the dimension of the system, and E_* defines the energy unit. The diffusion coefficient tends to zero exponentially fast as $I_*/I \to 0$ so that the diffusion time of an orbit increases exponentially as we move closer to the origin, thus defining the long-term dynamic aperture.

We are interested in the evolution of an initial distribution with an absorbing boundary condition and, in particular, in the estimate of the current at the boundary condition, including also the dependence on the system's parameters. Our approach is based on the stochastic differential equation that is associated to the Fokker-Planck equation (13), namely

$$dI = \frac{2\alpha E_*^2}{I_*}\left(\frac{I_*}{I}\right)^{\alpha+1} \exp\left[-2\left(\frac{I_*}{I}\right)^\alpha\right] d\tau + E_* \exp\left[-\left(\frac{I_*}{I}\right)^\alpha\right] dw_\tau \tag{15}$$

where w_τ is a Wiener process in the diffusion time τ.

Using Ito stochastic calculus, the change of variable

$$y = -\int_x^{x_a} \exp\left[\left(\frac{1}{x}\right)^\alpha\right] dx \qquad y \in [-\infty, 0] , \tag{16}$$

where $x = I/I_*$ and $x_a = I_{\mathrm{a}}/I_*$, reduces Eq. (15) to the form

$$dy = \alpha \left(\frac{E_*^2}{I_*}\right)^2 a(y)\, d\tau + \frac{E_*}{I_*} dw_\tau \tag{17}$$

with an absorbing boundary condition at $y = 0$ and

$$a(y) = \left[\frac{I_*}{I(y)}\right]^{\alpha+1} \exp\left[-\left(\frac{I_*}{I(y)}\right)^\alpha\right] . \tag{18}$$

We note that $a(y) > 0$ represents a drift field towards the boundary, which vanishes exponentially fast when $y \to -\infty$. moreover, the drift field is exponentially small if $I \ll I_*$. If $a(y)$ is well approximated by a linear field in the region of interest, one can apply an adiabatic approximation and compute the current at the boundary conditions by means of the fundamental solution of a Wiener process with an absorbing boundary.

To this aim, it is convenient to introduce an adimensional diffusion parameter $ds \propto (E_*/I_*)^2 d\tau$ to simplify Eq. (17). The diffusion parameter s is related to the physical time by the following relationship

$$ds = \left(\frac{\epsilon E_*}{I_*}\right)^2 \sqrt{T\gamma^{-1}} dt \,. \tag{19}$$

In the new variable y the Fokker-Planck equation reads

$$\frac{\partial \rho}{\partial s} = -\alpha \frac{\partial}{\partial y}\left[a(y)\rho\right] + \frac{1}{2}\frac{\partial^2 \rho}{\partial y^2} \,. \tag{20}$$

For an initial distribution $\delta(y + y_0)$ the probability current $\mathcal{J}(s|y_0)$ lost at the absorbing boundary reads

$$\mathcal{J}(s|y_0) \simeq \frac{y_0}{\sqrt{2\pi} s^{3/2}} \exp\left[-\frac{(\Phi^{-s}(y_0))^2}{2s}\right] , \tag{21}$$

where $\Phi^s(y_0)$ is the phase flow associated to the drift field $a(y)$. We derive a scaling law for the current (21) in the limit of null drift observing that if $y_0/\sqrt{s} \simeq 1$ then the probability current is of order $O(y_0^{-2})$ for a time $s \le y_0^2$. Indeed, letting $u = y_0/\sqrt{s}$ the expression (21) reads

$$\mathcal{J}(s|u) \simeq s\, u \exp\left(-\frac{u^2}{2}\right) . \tag{22}$$

The Nekhoroshev's estimate (10) for the perturbation implies

$$y_0 \simeq -\frac{C}{I_*} \exp\left(\frac{I_*}{I_0}\right)^\alpha$$

where C is a suitable constant. Then if $I_0 \ll I_*$ we get an exponentially small probability current for an exponentially long time and we recover an effective definition of dynamics aperture I_{da} (cfr. the scaling law (2))

$$I_{\mathrm{da}} \propto \frac{I_*}{(\ln s)^{1/\alpha}} \tag{23}$$

where the value of the diffusion parameter s is the related to stability time according to (19).

This result can be compared against the case of orbit diffusion with a power-law dependence of the perturbation strength, i.e.,

$$h(I) \propto \left(\frac{I}{I_*}\right)^{m+1} \tag{24}$$

for which the previous approach gives the estimate

$$y_0 \simeq \frac{1}{m\, I_*}\left(\frac{I_*}{I_0}\right)^m \tag{25}$$

so that the effective dynamic aperture is estimated to be

$$I_{\rm da} \propto \frac{I_*}{s^{1/(2m)}} , \tag{26}$$

which shows clearly that the scaling law is very different for the two models of diffusion coefficient.

Finally, for a generic initial distribution $\rho(I)$, the total probability current lost at the absorbing boundary is given by

$$\mathcal{J}(s) = \frac{1}{2\sqrt{2\pi} s^{3/2}} \int_0^{I_{\rm a}} y(I) \exp\left[-\frac{(\Phi_a^{-s}(I))^2}{2s}\right] \rho(I)\, d\, I\,, \tag{27}$$

which will be used to compare the analytical model with experimental data since Eq. (27) is related to the total beam losses, which are proportional to $1 - \mathcal{J}(s)$.

4. LHC dynamics aperture experiments

Measuring DA is a true challenge and it is an important goal in itself since it allows examination of non-linear single-particle motion.

In the method derived from Eq. (5), the beam can be blown-up gently in the transverse directions by means of an appropriate excitation until slow losses appear. Measurement of intensity decay with time then provides the needed information on the DA by performing a fit to the beam losses as a function of time. The proposed technique has been used in two experimental sessions at the LHC to probe the DA of Beam 1 at injection energy [28,29].

During the measurements, the collimation system is retracted to avoid any interference, and a single bunch is injected, gently blown-up transversely, and the intensity decay recorded. This is repeated for different values of the strength of the octupole circuits. These are special circuits (in total eight per ring) intended to compensate for the corresponding field error of the LHC main dipoles [13]. In the experimental sessions they have been used to shrink and control the DA.

Once a sizeable intensity change is obtained, the octupoles' strength is varied. This is repeated for several magnetic configurations of the LHC lattice. A summary of the experimental results is shown in Fig. 1, where the evolution of the relative beam intensity and of the transverse rms beam size are plotted for the various magnetic configurations analysed in this paper.

5. Analysis of the experimental intensity evolution

We have analysed the results of the LHC experimental sessions, as described in the previous section, to probe the models proposed in this paper. The key observable is the normalised beam intensity as a function of time, which has been compared against the predictions of the Fokker-Planck equation (13) in the diffusion time s and assuming an absorbing boundary condition at $I = I_{\rm a}$.

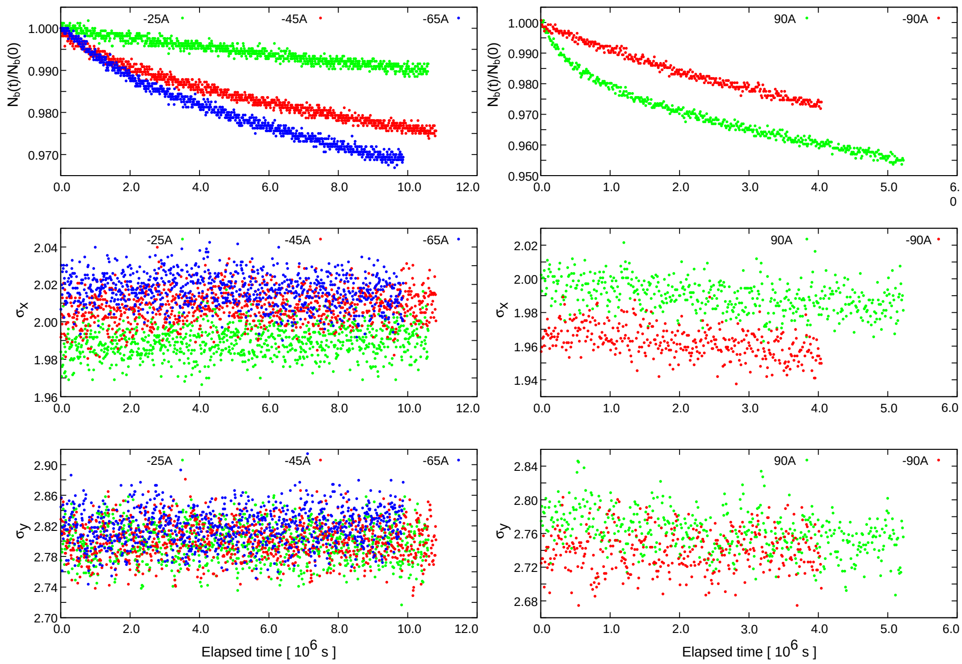

Fig. 1. Evolution of the relative beam intensity (upper), of the horizontal rms beam size (middle) and of the vertical rms beam size (lower) as a function of time for the DA measurement cases considered in this paper. The constancy of the beam size for the various cases is clearly seen

To obtain a closed-form analytic expression for the current at the boundary condition, we linearise the drift field, approximating it as a constant force by expanding $a(y)$ (cfr. Eq. (18)) at the initial condition, namely

$$a(y(I_0)) \simeq \left(\frac{I_*}{I_0}\right)^{\alpha+1} \exp\left[-\left(\frac{I_*}{I_0}\right)^{\alpha+1}\right]. \tag{28}$$

Since $a(I_0)$ is an increasing function for $I_0 \ll I_*$ we expect to underestimate the current at the absorbing barrier, but the effect of the drift field (18) is nevertheless small in the diffusion equation (20). Thus, the probability current $\mathcal{J}(s)$ at the boundary can be explicitly estimated to be

$$\mathcal{J}(s) = \frac{1}{2\sqrt{2\pi}s^{3/2}} \int_0^{I_\mathrm{a}} y(I_0) \exp\left[-\frac{(I_0 - a(I_0)s)^2}{2s}\right] \rho(I_0)\, d\, I_0\,. \tag{29}$$

In the experimental set-up, the initial distribution is well approximated by a Gaussian function in the physical variables, whose variance is the so-called beam emittance ε, which proven to be almost constant during the measurements. Since the unperturbed action I is well approximated by the linear invariant, i.e. the emittance, we assume an initial distribution $\rho(I_0)$ of the form

$$\rho(I_0) = \frac{1}{\varepsilon} \exp\left(-I_0/\varepsilon\right) . \tag{30}$$

Equation (29) can be used to obtain an explicit estimate for the relative intensity variation during the experiments, provided that the model's parameters are determined.

The position of the absorbing boundary I_{a} defines the action unit so that the free parameters are α, the scaling factor β between the physical time t and the diffusion time s, and I_*. To achieve a meaningful interpolation of the experimental data, we decided to reduce the number of free parameters by considering measurements referring to the same experimental set-up, i.e., with a single type of magnet being scanned in strength. Then, we assume that the exponent α depends only on the dimensionality and the non-linearities present in the system, hence it is kept the same for the whole set of experimental results. The scaling factor β depends on the ratio $(\epsilon/I_*)^2$ (cfr. Eq. (19)) and we assume that this ratio remains constant since both ϵ and I_* scale proportionally to the strength of the non-linear terms. Finally, given that ε was constant during the experiment it is possible to express I_{a} in terms of ε. As a consequence the only remaining free parameter in the interpolation procedure is I_*.

In Fig. 2 we show the results of interpolation of the experimental data using Eq. (29) for four values of the octupole currents. The values of the interpolation parameters are reported in Table 1 for all cases considered, using the position of the absorbing barrier I_{a} as the action unit. The interpolation results suggest a quite good agreement between the analytical approach and the experimental data in all the cases with a negative octupole current. This is obtained by varying only I_*, which is the estimate of the position of the fast escape to infinity. Note that α, β and $\varepsilon/I_{\mathrm{a}}$ are the same for all cases with negative octupole currents, whereas the best fit procedure provides a slightly varied exponent ($\alpha = 1.5$ instead of $\alpha = 1.4$) in the case of a positive octupole current of 90 A.

For the case with an octupole current of -90 A, we have compared the analytical estimate (29) with a direct integration of the Fokker-Planck equation for a much longer time than that considered in the experiments (see Fig. 3). We note that even after a time interval much longer than the typical elapsed time in the experiment, the relative error between the semi-analytical solution or the numerical one is less than 1%.

In Fig. 4 we report the relation between I_*/I_{a} (as derived from the interpolation procedure) and the octupole current. The continuous curve represents a linear interpolation of the data, which proves to be an excellent model of the data behaviour. This also confirms one of the hypothesis made in the determination of the theoretical model, i.e., that I_* scales with the non-linearities, which are, in turn, linear in the octupole current.

It is worthwhile stressing that the situation for the case with positive octupole

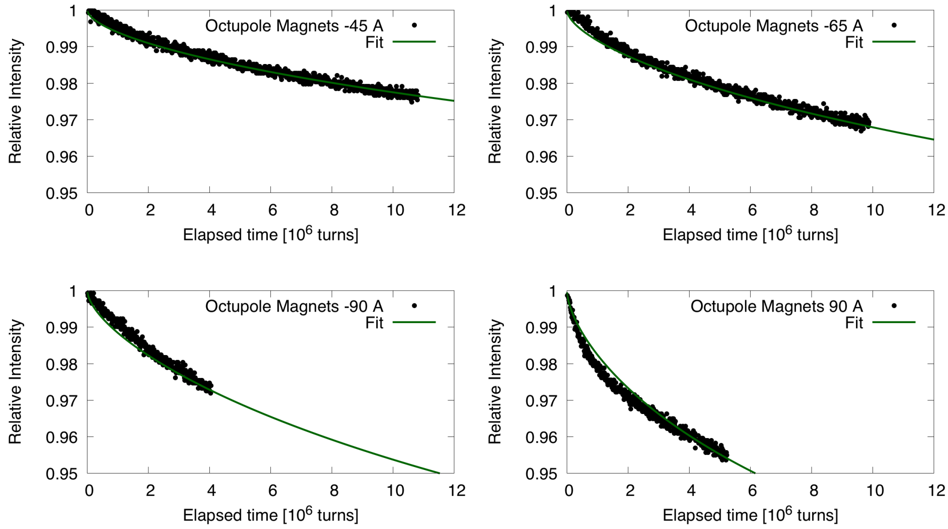

Fig. 2. Interpolation of the measured relative beam intensity during the LHC dynamic aperture experiments (black dots) using the analytic expression (29) (continuous curve) for four values of the octupole current, −45 A (upper left), −65 A (upper right), −90 A (lower left), and 90 A (lower right).

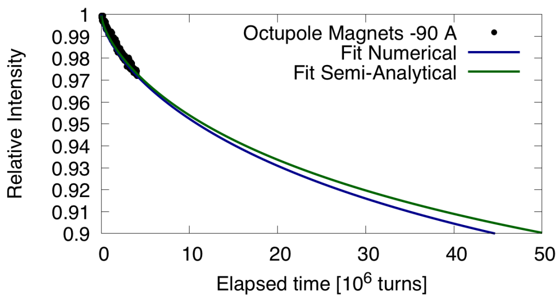

Fig. 3. Comparison between the fits of relative beam intensity for the case with −90 A octupole current (black dots) using the analytical estimate (29) (green curve) and the direct integration of the Fokker-Planck equation (13) with the same parameters (blue curve).

current looks rather different from the other data sets. In fact, even if the agreement between data and simulations is still reasonably good (see Fig. 2, lower right), the resulting parameters are different from those of the cases with negative currents. In particular, while the difference in terms of α is small, β is strongly reduced, meaning that the typical diffusion time is shorter than for the other cases. It is also worth observing that I_*/I_a for the case 90 A does not follow the same behaviour as the other cases as it does not follow the line shown in Fig. 4. This could be explained by considering that in the LHC ring the octupoles used in this experiment are not

Table 1. Parameter values obtained from the interpolation of the relative beam intensity.

Octupole Current [A]	I_*/I_a	α	β [10^4turns]	ε/I_a
-25	2.01	1.4	2.2	0.25
-45	1.62	1.4	2.2	0.25
-65	1.46	1.4	2.2	0.25
-90	1.30	1.4	2.2	0.25
+90	1.38	1.4	0.83	0.25

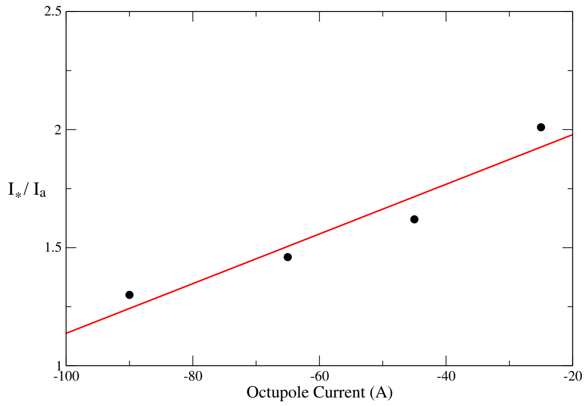

Fig. 4. Values of I_*/I_a computed by interpolating the experimental data with negative octupole currents. The linear interpolation curve has a correlation coefficient of 0.96 with the data.

the only source of non-linearities. Hence, a sign change of their strength produces a different effect in terms of compensation (or lack of compensation) with the other sources of non-linear effects present in the ring, thus introducing sharp changes in the interpolation parameters.

6. Conclusions

In this paper, we propose to estimate the long term DA in beam dynamics through a framework based on diffusion in stochastically perturbed Hamiltonian systems. The goal is to provide predictions that can be compared against experimental results. This was performed using experimental data from recent DA tests in the CERN LHC at injection energy and an analytical estimate of the loss current derived from a 1D Fokker-Planck equation for the action variable with an absorbing boundary condition and a Nekhoroshev-like diffusion coefficient. The preliminary results are encouraging, showing a good agreement between the predictions based on the diffusive models and the experimental results.

An essential next step to be accomplished consists of assessing in a clear way whether the form of the diffusion coefficient based on Nekhoroshev's estimate is really the best choice to interpolate the experimental data. This might require additional experimental sessions at the LHC that considers longer time series for the beam intensity.

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